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The Decay of Magnetic Fields

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Abstract

We discuss the decay of a system of currents in a (bounded) region of finite conductivity, assuming that the decay is very slow.

The chief results are a set of inter-related (vector) extremum principles for calculating the decay times associated with regions of arbtray shape and conductivity. These expressions differ according as the electric field vector, the magnetic field vector, or both, are permitted arbitrary variations.

The decay times are shown to be bounded from above; therefore, the values obtained from the variational principles approach the exact decay times from below, and the stationary property corresponds to the theorem that the exact field makes the decay time a maximum (Lenz's law).

A vector iterational procedure, generalizing the standard procedure in scalar problems, is discussed, and it is pointed out that two-dimensional decay problems also may be treated by conformal mapping.

A numerical application is the subject of a companion report.

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1. Introduction

Suppose a system of currents has been established in a certain region. If the conductivity is everywhere either zero or infinite, the fields persist indefinitely. However, if regions of finite conductivity are present, they eventually convert all the field energy into joulean heat.

We wish, in what follows, to consider such magnetic decay processes in the limit of large decay times. To make this restriction more precise, we first note that, if c is the velocity of light, τ the decay time, σ the conductivity (e.s.u.), and L a typical dimension of the region, then, as shown later,

$$\tau \sim \sigma \frac{L^2}{c^2} \quad . \tag{1.1}$$

Using (1.1), we may state our limiting condition in two equivalent forms: We shall restrict our discussion to situations for which

$$(c\tau/L)^2 \sim \sigma\tau \gg 1 \quad . \tag{1.2}$$

Physically, the first expression states that the decay times are very large compared with the time an electromagnetic disturbance requires to traverse the region; the second expression, applicable only where the conductivity is different from zero, states that the conduction current is much larger than the displacement current. Mathematically, (1.2) means that the simplified equations of the quasi-steady theory (no displacement current) are applicable.

Our interest in this problem stems from recent estimates, by Cowling[1] and Wrubel[2], of the decay times associated with stellar magnetic fields. Cowling, using a variational method, calculated the decay time characteristic of the lowest mode of the solar magnetic field. Wrubel, using numerical integration, computed decay times for several higher modes. The present remarks start from the observation that Cowling's variational method may be generalized to apply to an arbitrary vector mode, or indeed to an arbitrary vector field.

In what follows, we give a systematic development of a vector formalism suitable for the calculation of decay times associated with regions of arbitrary shape and conductivity. Our chief results are a set of interrelated extremum principles for the decay time. These expressions differ from one another according as the electric field vector, the magnetic field vector, or both, are permitted to undergo arbitrary variations about their correct values. Since, as we shall show, the decay times are bounded from above, our variational principles approach the exact decay times from below. This fact may be expressed as a theorem: the exact field makes the decay time a maximum.

A vector iterational procedure, for use with the variational principles, can be set up quite simply; such a scheme, generalizing the standard procedure in scalar problems, is described. As a final remark, it is pointed out that two-dimensional decay problems may be treated by conformal mapping procedures.

In this report, we give a general discussion of the mathematical formalism, describing the variational principles and the iterational procedure in detail. In a companion report[3], which may be read independently, one of the variational expressions developed here is derived directly and is applied to the problem considered by Wrubel. Comparison with Wrubel's numerical integration shows that a simple trial function approximates the exact result quite closely.

2. The problem

We consider a set of currents in a body of conductivity $\sigma(\mathbf{r})$. The currents, and the fields associated with them, decay according to $e^{-t/\tau}$. We wish to calculate τ.

We shall be interested only in the quasi-steady case. That is, (1.2) applies. Then, from the first equation below, $\underline{E} \sim \underline{H}\,(L/c\tau)$; in the second equation, therefore, the displacement current, $-\frac{E}{c\tau}$, can be omitted, since it is of order $(L/c\tau)^2$ compared with the curl $\underline{H}$ term.

Thus, the field equations, in Gaussian units, are ($\varepsilon = \mu = 1$):

$$\nabla \times \underline{E} = \underline{H}/c\tau \tag{2.1}$$

$$\nabla \times \underline{H} = \frac{4\pi}{c}\,\sigma\,\underline{E} \tag{2.2}$$

where the last equation will also be valid outside the body, within a region of dimension $\sim L$, if we put $\sigma = 0$.

We shall also require that

$$\nabla \cdot \underline{H} = 0 \tag{2.3}$$

$$\nabla \cdot \underline{E} = 0\,. \tag{2.4}$$

The first of these equations is a consequence of (2.1); the second, which states that the charge density is always zero, is closely related to (1.2); for, if $\sigma\tau \gg 1$, the volume charge decays in a time that is very small compared to the decay time.

A further condition follows from Eqs. (2.2) and (2.4):

$$\underline{E} \cdot \nabla \sigma = 0\,. \tag{2.5}$$

For example, if σ is spherically symmetric, $\underline{E}$ will not have a radial component (H-modes only).

Because Eqs. (2.1) and (2.2) are valid only within lengths $\sim L$, it is convenient to limit our analytical manipulations to some region of maximum dimension $< L$. The conducting body, assumed bounded, itself suggests such a region. The influence of the fields outside the body is represented by imposing a boundary condition on the surface, S, of the body. Using the (symmetric) dyadics, $\mathcal{Z}$ and $\mathcal{Y}$, and the unit normal, $\underline{n}$, pointing out of the body, the boundary condition can be given in either of the two equivalent forms

$$\left.\begin{aligned} &(2.6) \qquad \underline{n} \times \underline{E} = \mathcal{Z} \cdot H \\ &(2.7) \qquad \underline{H} \times \underline{n} = \mathcal{Y} \cdot E \end{aligned}\right\} \text{ on S.}$$

Here, the (surface) dyadics $\mathcal{Y}$ and $\mathcal{Z}$ are related by:

$$(2.8) \qquad \mathcal{Y} = -\underline{n} \times \mathcal{Z}^{-1} \times \underline{n}; \qquad \mathcal{Z} = -\underline{n} \times \mathcal{Y}^{-1} \times \underline{n}.$$

Outside S, both E and H satisfy Laplace's equation.

Poynting's theorem follows from Eqs. (2.1) and (2.2):

$$(2.9) \qquad \nabla \cdot (\underline{E} \times \underline{H}) = \frac{H^2}{c\tau} - \frac{4\pi}{c} \sigma \underline{E}^2$$

as do the additional identities:

$$(2.10) \qquad \begin{aligned} \frac{H^2}{c\tau} + \frac{4\pi}{c} \sigma \underline{E}^2 &= \underline{H} \cdot \nabla \times \underline{E} + \underline{E} \cdot \nabla \times \underline{H} \\ &= \nabla \cdot (\underline{E} \times \underline{H}) + 2\, \underline{E} \cdot \nabla \times \underline{H} \\ &= \nabla \cdot (\underline{H} \times \underline{E}) + 2\, \underline{H} \cdot \nabla \times \underline{E}. \end{aligned}$$

In the following, we always integrate over the conducting region (volume V, surface S). For example, from (2.9),

$$(2.11) \qquad c\tau = \frac{\int dV\, H^2}{\frac{4\pi}{c} \int dV\, \sigma \underline{E}^2 + \int dS\, \underline{n} \cdot \underline{E} \times \underline{H}}\,.$$

This equation is a field generalization of the definition of the decay constant of a lumped constant circuit: for a simple R, L circuit, the decay time is ~ L/R; in our case, (2.11) defines τ as the ratio of the stored magnetic energy to the energy dissipated per second.

Although τ may be computed from (2.11), the stationary properties of this expression are not clear. These properties, which are more conveniently based on the identities (2.10), will now be discussed.

3. Variational expressions[*]

Eqs. (2.1) and (2.2), and the boundary condition (2.6), may be derived by demanding that the expression below be stationary when E and H are varied independently:

$$(3.1)\qquad L_1 = \frac{1}{2}\int dV\left\{\frac{\underline{H}^2}{c\tau} + \frac{4\pi}{c}\sigma\underline{E}^2 - 2\underline{E}\cdot\nabla\times\underline{H}\right\} - \frac{1}{2}\int dS\ \underline{H}\cdot\mathcal{Z}\cdot\underline{H}$$

If the boundary condition is used in the form (2.7), the Lagrangian above is replaced by:

$$(3.2)\qquad L_2 = \frac{1}{2}\int dV\left\{\frac{\underline{H}^2}{c\tau} + \frac{4\pi}{c}\sigma\underline{E}^2 - 2\underline{H}\cdot\nabla\times\underline{E}\right\} + \frac{1}{2}\int dS\ \underline{E}\cdot\mathcal{Y}\cdot\underline{E}\ .$$

We now verify the stationary properties of L_1 and L_2:

$$\delta L_1 = \int dV\left\{\frac{\underline{H}\cdot\delta\underline{H}}{c\tau} + \frac{4\pi}{c}\sigma\underline{E}\cdot\delta\underline{E} - \delta\underline{E}\cdot\nabla\times\underline{H} - \underline{E}\cdot\nabla\ \ \delta\underline{H}\right\} - \int dS\ \delta\underline{H}\cdot\mathcal{Z}\cdot\underline{H}$$

The last term in the volume integral may be rewritten:

$$(3.3)\qquad -\underline{E}\cdot\nabla\times\delta\underline{H} = \nabla\cdot(\underline{E}\times\delta\underline{H}) - \delta\underline{H}\cdot\nabla\times\underline{E},$$

so that:

$$(3.4)\qquad \delta L_1 = \int dV\left\{\delta\underline{H}\cdot\left[\frac{\underline{H}}{c\tau} - \nabla\times\underline{E}\right] + \delta\underline{E}\cdot\left[\frac{4\pi}{c}\sigma E - \nabla\times\underline{H}\right]\right\} + \int dS\ \delta\underline{H}\cdot\left[n\times\underline{E} - \mathcal{Z}\cdot\underline{H}\right].$$

---

[*] The Lagrangians used in this sections were suggested by analogous forms, for the oscillatory case, discussed by J. Schwinger in war-time lectures at the M.I.T. Radiation Laboratory, and by N. Marcuvitz in recent lectures at New York University.

Similarly,

$$(3.5) \qquad \delta L_2 = \int dV \left\{ \delta H \cdot \left[\frac{\underline{H}}{c\tau} - \nabla \times \underline{E} \right] + \delta \underline{E} \cdot \left[\frac{4\pi}{c} \sigma \underline{E} - \nabla \times \underline{H} \right] \right\}$$

$$- \int dS \, \delta \underline{E} \cdot \left[\underline{H} \times \underline{n} - \mathcal{Y} \cdot \underline{E} \right] .$$

We conclude from these equations that for independent variations of $\underline{E}$ and $\underline{H}$ about the values that satisfy Eqs. (2.1), (2.2), and (2.6), L_1 is stationary; conversely, if L_1 is stationary for independent variations of $\underline{E}$ and $\underline{H}$, Eqs. (2.1), (2.2), and (2.6) follow. A corresponding statement holds for L_2 if Eq. (2.6) is replaced by (2.7). Also, from the identities (2.10), the correct fields - and boundary conditions - make the stationary values of both L_1 and L_2 zero. Thus, if L_1 and L_2 are each equated to zero, we get two variational expressions for the decay time, τ.

In these variational expressions, both $\underline{E}$ and $\underline{H}$ can be varied independently. As an alternative, we may use one Maxwell equation to define one of the field vectors in terms of the other. Putting this into our variational principles gives us two new variational expressions, each involving only a single field vector. Varying this vector, and imposing the stationary property, yields the other Maxwell equation and the boundary condition.

For example, we may define $\underline{E}$ in terms of $\underline{H}$ by Eq. (2.2). Putting this in L_1, we get L_1':

$$(3.6) \qquad L_1' = \frac{1}{2} dV \left\{ \frac{H^2}{c\tau} - \frac{(\nabla \times \underline{H})^2}{\frac{4\pi\sigma}{c}} \right\} - \frac{1}{2} \int dS \, \underline{H} \cdot \mathcal{Z} \cdot \underline{H}.$$

Using (3.3), one easily verifies that $\delta L_1' = 0$ for variations of $\underline{H}$ about (2.1) and (2.6). Since $L_1' = 0$ for the correct choice of $\underline{H}$, (3.6), set equal to zero, is a variational principle for τ in which only the magnetic field is varied.

A similar variational principle for τ, that involves only the electric field, is obtained by defining $\underline{H}$ in terms of $\underline{E}$ by Eq. (2.1); putting this into L_2; we find:

$$L_2' = \frac{1}{2}\int dV \left\{ \frac{4\pi}{c}\,\sigma \underline{E}^2 - c\,\tau(\nabla\times\underline{E})^2 \right\} + \frac{1}{2}\int dS\, \underline{E}\cdot\mathcal{Y}\cdot\underline{E}\,. \tag{3.7}$$

It now follows that $L_2' = 0$ if E satisfies Eqs. (2.2) and (2.7); also $\delta L_2' = 0$ if $\underline{E}$ is varied about the solution of these equations. Hence, (3.7), set equal to zero, is a variational principle for τ that involves electric field variations only.

Using Eqs. (2.6) and (2.7), we note that:

$$\underline{n}\cdot\underline{E}\times\underline{H} = \underline{H}\cdot\mathcal{Z}\cdot\underline{H} = \underline{E}\cdot\mathcal{Y}\cdot\underline{E} \tag{3.8}$$

The stationary properties of (2.11) are now clear: $\underline{E}$ may be eliminated from (2.11) by using (2.2) in the volume integral and (3.8) in the surface integral; the result is identical with the variational principle obtained by putting $L_1' = 0$. Similarly, H may be eliminated from (2.11) by using (2.1) in the volume integral and (3.8) in the surface integral, the result now being the variational principle obtained by setting $L_2' = 0$.

From an examination of (3.4) and (3.5), we see that the sole function of the surface terms in (3.1) and (3.2) is to insure that the variations are made about fields that satisfy (2.6) and (2.7). Correspondingly, in (3.6) the surface term limits the allowable variations of H: the magnetic field must be varied about the field that satisfies

$$\underline{n}\times\left(\frac{\nabla\times\underline{H}}{\frac{4\pi\sigma}{c}}\right) = \mathcal{Z}\cdot\underline{H} \qquad \text{(on S)} \tag{3.9}$$

A like statement holds for (3.7): because of the surface term, the variations of $\underline{E}$ must be made about the field that satisfies (we have set $\mathcal{Y} = c\tau\mathcal{Y}'$):

$$(3.10) \qquad (\nabla \times \underline{E}) \times \underline{n} = \mathcal{Y}' \cdot \underline{E} \qquad \text{(on S)}$$

It follows, then, that if we impose the restrictions (3.9) and (3.10) separately from the variational principles, the surface terms in $L_1' = 0$ and $L_2' = 0$ may be eliminated.

For example, setting $L_1' = 0$, and using (2.2), (3.8), and the identity

$$\frac{(\nabla \times \underline{H})^2}{\frac{4\pi\sigma}{c}} = \nabla \cdot \left\{ \frac{\underline{H} \times (\nabla \times \underline{H})}{\frac{4\pi\sigma}{c}} \right\} + \underline{H} \cdot \nabla \times \left\{ \frac{\nabla \times \underline{H}}{\frac{4\pi\sigma}{c}} \right\},$$

we find:

$$(3.11) \qquad c\tau = \frac{\int dV\, \underline{H}^2}{\int dV\, \underline{H} \cdot \nabla \times \left\{ \frac{\nabla \times \underline{H}}{\frac{4\pi\sigma}{c}} \right\}} \,.$$

(3.11) is stationary for variations of H about the field that satisfies (3.9) on the surface of the conducting body and

$$(3.12) \qquad \nabla \times \left\{ \frac{\nabla \times \underline{H}}{\frac{4\pi\sigma}{c}} \right\} = \frac{\underline{H}}{c\tau}$$

in the interior.

If we set $L_2' = 0$, we may use (2.1) and (3.8) to get:

$$(3.13) \qquad c\tau = \frac{\int dV\, \frac{4\pi\sigma}{c} \underline{E}^2}{\int dV\, \underline{E} \cdot \nabla \times (\nabla \times \underline{E})} \,.$$

This expression is stationary for variations of $\underline{E}$ about the field that satisfies (3.10) on S and

(3.14) $$\nabla \times (\nabla \times \underline{E}) = \frac{4\pi\sigma}{c^2\tau}\underline{E}$$

inside S.

We conclude this section with a brief discussion of the spectrum of τ. We put

(3.15) $$\lambda = \frac{4\pi}{c^2\tau}$$

and, as required by (1.2), we impose (2.4) on (3.14). The result is:

(3.16) $$(\nabla^2 + \lambda\sigma)\,\underline{E} = 0,$$

where E must be represented by its rectangular components. We now wish to show that the spectrum of τ (as determined by the differential equation (3.16) and the boundary condition (3.10)) is bounded from above; for, if this is so, our variational expressions for τ will always approximate τ from below.

We shall sketch a proof based on quantum-mechanical reasoning: Put

(3.17) $$\frac{4\pi\sigma}{c^2\tau} = \lambda\sigma = \frac{-2m}{\hbar^2}\,V.$$

Then (3.16) becomes, for each component of $\underline{E}$, a Schrödinger equation that describes a particle of mass m bound, with zero binding energy, to a potential V. (Since $\lambda\sigma$ is always > 0, V is an attractive potential.) Thus, the equivalent quantum-mechanical problem concerns a potential of fixed range (L) and shape (σ); the scale factor of the potential, that leads to zero binding energy, is the eigenvalue that must be determined.

Since the particle energy is zero, and V is always < 0, the magnitudes of the kinetic energy, T, and the potential energy, V, are always equal: That is,

$$|V| = T = \frac{p^2}{2m},$$

where P is the particle momentum. On the other hand, because of the uncertainty principle, P is always of the order of, or greater than, $\hbar/L$. Therefore

$$\frac{2m}{\hbar^2} \; |V| \; L^2 \gtrsim 1$$

or, returning to our decay problem via (3.17),

$$\tau \lesssim \frac{4\pi\sigma L^2}{c^2} \tag{3.18}$$

Thus, τ is bounded from above*, and the values of τ given by using approximate fields in the variational principles will always be less than the true value of τ.

These remarks may be expressed as a thoerem: the exact field makes the decay time a maximum. In this form, we have what appears to be an example of Lenz's law: namely, the fields adjust themselves to oppose any change.

Incidently, we note that (3.18) justifies the estimate of τ given in (1.1).

4. Iteration of E

The variational expressions given above are most useful when combined with some systematic method of improving the trial functions to be used. Such an iterational scheme is well known[4] for scalar variational principles of the type we have discussed. We wish to show, in this section, that an iterational procedure may also be set up for the vector variational principles of Section 3.

A simple way of accomplishing this is merely to generalize the usual scalar treatment to apply to vector functions. Thus, we define the sequence of functions E_k by the conditions:

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* The argument may be extended to show that the spectrum of τ is also discrete.

$$\nabla \cdot \underline{E}_k = 0 \tag{4.1}$$

$$\nabla \times (\nabla \times \underline{E}_k) = -\nabla^2 \underline{E}_k = \frac{4\pi}{c^2} \sigma \underline{E}_{k-1} \tag{4.2}$$

$$(\nabla \times \underline{E}_k) \times \underline{n} = \mathcal{Y}' \cdot \underline{E}_k \qquad \text{(on S).} \tag{4.3}$$

In the second equation, the Laplacian operates on the rectangular components of $\underline{E}_k$.

If we apply this iterational scheme to Eq. (3.13), we can show that, as k → ∞, the trial functions, $\underline{E}_k$, and the corresponding values of τ, τ_{2k}, approach the exact eigenfunctions and eigenvalues.

The proof follows in close analogy with the scalar case: First, using (4.3), we note that:

$$\int dS\, \underline{E}_\ell \cdot \mathcal{Y}' \cdot \underline{E}_k = \int dV \left\{ (\nabla \times \underline{E}_\ell) \cdot (\nabla \times \underline{E}_k) - \underline{E}_\ell \cdot \nabla \times (\nabla \times \underline{E}_k) \right\}.$$

The left side of this equation, and the first term on the right, are symmetrical in ℓ and k. Therefore, the remaining term on the right must also be unaffected by interchanging ℓ and k. That is, introducing $(\ell + k - 1)$ as defined below,

$$(\ell + k - 1) = \int dV\, \sigma \underline{E}_\ell \cdot \underline{E}_{k-1} = \int dV\, \sigma \underline{E}_{\ell-1} \cdot \underline{E}_k\,, \tag{4.4}$$

the integral in (4.4) depends only on the sum of the indices.

If we put $\underline{E}_k$ into (3.13) and call the resultant approximation τ_{2k}, we find:

$$\tau_{2k} = \frac{(2k)}{(2k-1)}\,. \tag{4.5}$$

We now show that the sequence τ_{2k} increases with k. The proof proceeds in two stages: From

$$\int dV \, \sigma (\alpha \underline{E}_k + \underline{E}_{k+1})^2 \geq 0$$

we find

$$\frac{1}{(2k)} \left\{ (2k)\alpha + (2k+1) \right\}^2 + (2k+2) - \frac{(2k+1)^2}{(2k)} \geq 0 .$$

Since this holds for all α, it follows that

$$\frac{(2k+1)}{(2k)} \leq \frac{(2k+2)}{(2k+1)} ; \quad \text{or:} \quad \tau_{2k+1} \leq \tau_{2k+2} . \tag{4.6}$$

Next, we note that

$$\int dS \, \underline{n} \cdot \underline{E} \times \underline{H} = \int dS \, \underline{E} \cdot \mathcal{Y} \cdot \underline{E} = \int dS \, \underline{H} \cdot \mathcal{Z} \cdot \underline{H} \leq 0$$

for any fields that satisfy the boundary conditions. This follows from the interpretation of the Poynting vector as the flux of energy: for, since the energy is dissipated only inside the conducting region, the energy in the external region must have a net flow in the direction opposite to $\underline{n}$ (the outward normal to S). Also, since the boundary conditions are derived by considering the exact fields in the external region, the boundary conditions will insure that the energy flow is in the correct direction.

We may therefore write, since $\mathcal{Y}$ and $\mathcal{Y}'$ differ only by a positive constant,

$$\int dV \left\{ \nabla \times (\alpha \underline{E}_k + \underline{E}_{k+1}) \right\}^2 - \int dS (\alpha \underline{E}_k + \underline{E}_{k+1}) \cdot \mathcal{Y}' \cdot (\alpha \underline{E}_k + \underline{E}_{k+1}) \geq 0 ,$$

or, using (4.3),

$$\int dV \, \sigma (\alpha \underline{E}_k + \underline{E}_{k+1}) \cdot (\alpha \underline{E}_{k-1} + \underline{E}_k) \geq 0 .$$

From this it follows that:

$$(4.7) \qquad \frac{(2k)}{(2k-1)} \leq \frac{(2k+1)}{(2k)} \text{ ; or: } \qquad \tau_{2k} \leq \tau_{2k+1} .$$

In summary, we have

$$(4.8) \qquad \tau_{2k} \leq \tau_{2k+1} \leq \tau_{2k+2} \cdots\cdots \leq \tau .$$

The remainder of the convergence proof proceeds exactly as in the scalar case: if $\tau^{(1)}$ is the largest eigenvalue, and $\underline{E}^{(1)}$ is the corresponding eigenfunction, then

$$\tau^{(1)}_{2k} \rightarrow \tau^{(1)}$$

$$\underline{E}^{(1)}_{k} \rightarrow \underline{E}^{(1)},$$

as $k \rightarrow \infty$, providing that the initial (generating) function is not orthogonal to $\underline{E}^{(1)}$. A corresponding statement holds for the smaller eigenvalues $\tau^{(n)} (n \neq 1)$ and their eigenfunctions $\underline{E}^{(n)}$, except that now the generating function is chosen orthogonal to all $\underline{E}^{(k)}$ for $k < n$.

An essential part of the convergence proof is a statement concerning the boundedness of each member of the sequence of iterated functions. For this purpose, it is convenient to replace Eqs. (4.2) and (4.3) by an integral relation. This may be done by defining a (scalar) Green's function to satisfy:

$$(4.9) \qquad \nabla^2 G(\underline{r}, \underline{r}') = -\delta(\underline{r} - \underline{r}'),$$

$$(4.10) \qquad (\nabla G(\underline{r}, \underline{r}') \times I) \times \underline{n} = \mathcal{Y}' G(\underline{r}, \underline{r}') \qquad (\underline{r} \text{ on } S),$$

where I is the unit dyadic. We then have:

$$(4.11) \qquad \underline{E}_k(\underline{r}) = \int dV' G(\underline{r}, \underline{r}') \, \frac{4\pi}{c^2} \, \sigma(\underline{r}') \, \underline{E}_{k-1}(\underline{r}')$$

as the solution of (4.2) subject to the boundary condition (4.3).

5. Iteration of $\underline{H}$

So far, we have considered only the iteration of $\underline{E}$. The iteration of $\underline{H}$ appears to be more difficult. One method, however, is to make the iteration of $\underline{H}$ dependent on that of $\underline{E}$ by defining:

$$(5.1) \qquad \underline{H}_k = \nabla \times \underline{E}_k ,$$

where $\underline{E}_k$ is fixed by Eqs. (4.1)-(4.3). We now have from (4.2),

$$(5.2) \qquad \nabla \times \left\{ \frac{\nabla \times \underline{H}_k}{\frac{4\pi\sigma}{c}} \right\} = \underline{H}_{k-1} \; .$$

The boundary condition on $\underline{H}_k$ is automatically fixed by that on $\underline{E}_k$; from (4.3) and (5.1)

$$\underline{H}_k \times \underline{n} = \mathcal{Y}' \cdot \underline{E}_k = \mathcal{Y}' \cdot \left\{ \frac{\nabla \times \underline{H}_{k+1}}{\frac{4\pi\sigma}{c}} \right\} \qquad \text{(on S).}$$

Using $\mathcal{Y} = c\tau\mathcal{Y}'$ and (2.8), and defining $\mathcal{Z}' = c\tau\mathcal{Z}$, the equation above is equivalent to:

$$(5.3) \qquad \underline{n} \times \left\{ \frac{\nabla \times \underline{H}_{k+1}}{\frac{4\pi}{c}\sigma} \right\} = \mathcal{Z}' \cdot \underline{H}_k \qquad \text{(on S).}$$

In Eqs. (5.2) and (4.3), the iteration of $\underline{H}$ has been freed of any explicit dependence on the iteration of $\underline{E}$. However, it appears from the derivation of these equations that the two iteration schemes are not independent, but each is equivalent to the other.

6. Mapping

Two-dimensional decay problems are amenable to treatment by conformal mapping methods. For example, taking Eqs. (3.16) in two dimensions, suppose that we map the z-plane ($z = x + iy$) into the z'-plane ($z' = x' + iy'$):

$$z = z(z'). \tag{6.1}$$

The line elements are related by (we use * for complex conjugate):

$$dx^2 + dy^2 = dz\, dz^* = \left|\frac{dz}{dz'}\right|^2 (dx'^2 + dy'^2)$$

Therefore, in the z'-plane, we get an equation identical in form to (3.16) except that $\sigma(z)$ is replaced by

$$\sigma' = \left|\frac{dz}{dz'}\right|^2 \sigma. \tag{6.2}$$

By means of (6.1) and (6.2), the conductivities of two problems having the same eigenvalue, λ, may be related to one another.

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